

## FINITE ELEMENTS IN LINEAR VISCOELASTICITY

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A stress analysis method is presented which includes the capability for transient, non homogeneous temperature distribution. The method is based on the following assumptions: linear viscoelasticity with hereditary integral form of stress-strain relation; validity of the reduced time hypothesis; bulk modulus constant in time; homogeneous, isotropic material. The associated, finite element computer program, called TAP (Thermoviscoelastic Analysis Program), is a plane strain formulation with uniform strain and linear temperature distribution assumed in each element. The element matrices involve superposition integrals which are approximated numerically for stepping out the time varying solution. TAP solutions for simple problems are compared with exact and other approximate solutions and with other approximate methods to more complex problems. Different numerical procedures are considered, in particular one that avoids integration back to the time origin thus minimizing computer storage and solution times.

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## SECTION I

### INTRODUCTION

This paper was motivated by problems of solid propellant grain stress analysis that arise under transient thermal environment. At present, most stress analyses of rocket grains use an elastic analysis which by various techniques is made to approximate the viscoelastic character of the solid propellant response.

One such technique uses elastic moduli equal to the corresponding viscoelastic properties at the desired time and temperature. This method, commonly referred to as quasi-elastic, essentially ignores the entire past history of loading and environment and may be a gross approximation to the true response.

A generally more preferable technique is based on the correspondence principle, or elastic-viscoelastic analogy, first elucidated by Alfrey (Reference 1) which permits one, using an elastic solution, to obtain the corresponding viscoelastic solution to the same geometric problem with equivalent time-step boundary conditions. The procedure is exact if a closed form elastic solution exists and if the inverse of the Laplace transformed viscoelastic solution can be obtained exactly. For use with numerical solutions, Schapery, (Reference 2) has shown how the method is applied as an approximate inverse Laplace transform technique. The technique is appropriate where the required correspondence exists between the elastic and transformed viscoelastic problem.

As pointed out by others, Morland and Lee (Reference 3), the correspondence principle fails to hold for nonhomogeneous, transient temperature distribution, although still valid for either homogeneous, transient temperature or nonhomogeneous, steady state temperatures. The difficulty involves one of two things, depending on whether the problem is formulated in real time or shifted time. In terms of real time, the stress-strain law is not a convolution integral; if the shifted time is used, the transformed viscoelastic equilibrium and strain displacement equations are not equivalent to the corresponding elastic equations.

To circumvent these problems, Hilton and Russell (Reference 4) impose conditions of constant temperature over time increments and apply the correspondence principle on an incremental basis. This method has been used successfully by Valanis and Lianis (Reference 5) although its application becomes somewhat involved.

Regardless of what indirect means are used to obtain a solution to the thermoviscoelastic problem, one is still faced with assessing the accuracy of the solution, given a particular viscoelastic material, history of load and environment, and solution time realm. Therefore a need exists for a direct method of attack on the problem - one in which there is confidence of an accurate solution if a fine enough time grid is used and general enough to include two dimensional, irregular geometries and arbitrary material properties.

A first step in this direction was taken by Taylor and Chang (Reference 6) who formulated the one dimensional plane strain circular cylinder problem by a finite element procedure, then demonstrated its application to steady state temperature fields. As they point out, computer time can quickly become excessive for large problems, which is due partly by the need for excessive out-of-core storage. The solution procedure involves generation and solution of a new set of stiffness equations for each time step. The solutions are saved for all time steps and used to generate new load columns which involve summation back to the time origin, thus requiring a great amount of storage for long-time realms and large numbers of freedoms.

The present paper is a two dimensional, plane strain, finite element formulation of the thermoviscoelastic problem, in which steps are taken to overcome some of the aforementioned difficulties. It is a direct approach, not relying on equivalent elastic solutions or transform methods.

The assumption of time-constant bulk modulus is made (an apparent good assumption for some solid propellant materials for example, Reference 7) which leads to a particularly simple time-varying form of the stiffness matrix and minimizes regeneration of matrices at each time step. Also, it is shown how the history or memory effects can be formulated so that the memory load (a generalized load which accounts for the history of deformation) may be expressed as a recurrence relation from one time step to the next rather than requiring summation to the time origin.

## SECTION II

## ONE DIMENSIONAL PROTOTYPE EQUATION

Certain aspects of the general equations in the next section can be illustrated in one dimension. We therefore consider briefly the numerical inversion of the uniaxial, viscoelastic stress-strain relation,

$$\sigma(t) = E(0)\epsilon(t) - \int_0^t \frac{dE(t-t')}{dt'} \epsilon(t') dt' \quad (1)$$

where  $\sigma(t)$  and  $\epsilon(t)$  are uniaxial stress and strain,  $E(t)$  is the uniaxial stress relaxation modulus defined as the stress response to a unit step strain. In Equation 1 explicit allowance for an initial step strain has been made. When  $\sigma(t)$  is prescribed and  $E(t)$  is given as a tabulated numerical function, it is desired to find a numerical solution,  $\epsilon(t)$ . When  $\sigma(t)$  is a unit step, the solution  $\epsilon(t)$  is the uniaxial creep compliance,  $D(t)$ . The numerical form of Equation 1 using a trapezoidal approximation is

$$\sigma(t_k) = E(0)\epsilon(t_k) - \sum_{i=1}^{k-1} \frac{\epsilon(t_{i+1}) + \epsilon(t_i)}{2} [E(t_k - t_{i+1}) - E(t_k - t_i)] \quad (2)$$

or rearranging,

$$\begin{aligned} \left[ \frac{E(0) + E(t_k - t_{k-1})}{2} \right] \epsilon(t_k) &= \sigma(t_k) + \frac{\epsilon(t_{k-1})}{2} [E(0) - E(t_k - t_{k-1})] \\ &+ \sum_{i=1}^{k-2} \epsilon^*(t_i) [E(t_k - t_{i+1}) - E(t_k - t_i)] = \sigma(t_k) + V(t_k) \end{aligned} \quad (3)$$

$$k = 1, 2, 3, \dots \quad t_1 = 0 \quad \epsilon^*(t_i) = \frac{\epsilon(t_{i+1}) + \epsilon(t_i)}{2}$$

from which is determined successively  $\epsilon(t_k)$ . The "memory" term,  $V(t_k)$ , which involves the past solutions, is known. Lee and Rogers (Reference 8) have discussed convergence using the trapezoidal approximation.

A simpler approximation is to assume  $E(t)$  constant over each time step, a staircase function falling below the exact  $E(t)$ . The advantage of this approximation is that an upper value for the time varying solution (creep compliance) is obtained, which can be demonstrated analytically. The trapezoidal approximation on the other hand appears to give a value below the exact solution. This information could possibly be used to bracket the solution to certain types of problems.

In order to gain some insight into the numerical requirements for differing viscoelastic materials, two representative relaxation functions were inverted, one a very simple, standard linear solid representation and the other a more complex, broad band representation. The results are shown in Figures 1 and 2. The inversion for the simpler material converged rapidly but the reflected curve  $1/E(t)$  deviated greatly from  $D(t)$ . The more complex representation, for which a closed form exact creep compliance is unobtainable, showed much slower convergence but an apparently reasonable representation of  $D(t)$  by  $1/E(t)$ . We might therefore expect quasi-elastic solutions to be acceptable engineering approximation in some cases.

Ordinarily the summation in Equation 2 must be carried out back to the time origin requiring that every previous solution be saved. In the multi-dimensional case considered in the next section, vector solutions must be saved and summed at each time step so that storage requirements and perhaps computer running time can become critical for large  $t$ . There are ways to avoid this problem, none of which in general appear to be completely satisfactory.

One way is to truncate the summation for  $i \ll k$  when the terms  $[E(t_k - t_i) - E(t_k - t_{i-2})] \epsilon(t_{i-1})$  become negligibly small since  $E(t)$  is a monotonically decreasing function. In the absence of temperature effects  $[E(t_k - t_i) - E(t_k - t_{i-2})]$  generally decreases rapidly; however, a past history of relatively large deformation could offset this so that care must be exercised. Further, the effect of low temperature is to retard the rate of decrease of  $E(t)$ .

Another way of avoiding the storage requirement is to replace the summation by a recurrence relation. Zak (Reference 9) has shown how this can be done by expressing the relaxation modulus as a Prony series as follows:

$$E(t) = A_0 + \sum_{j=1}^{j=q} A_j e^{-\frac{t}{\tau_j}} \quad (4)$$

The Expression 2 then becomes

$$\begin{aligned} \sigma(t_k) = & E(0) \epsilon(t_k) - \frac{\epsilon(t_k) + \epsilon(t_{k-1})}{2} [E(0) - E(t_k - t_{k-1})] \\ & - \sum_{i=1}^{k-2} \epsilon^*(t_i) \left[ \sum_{j=1}^q A_j \left( e^{-\frac{(t_k - t_{i+1})}{\tau_j}} - e^{-\frac{(t_k - t_i)}{\tau_j}} \right) \right] \end{aligned} \quad (5)$$

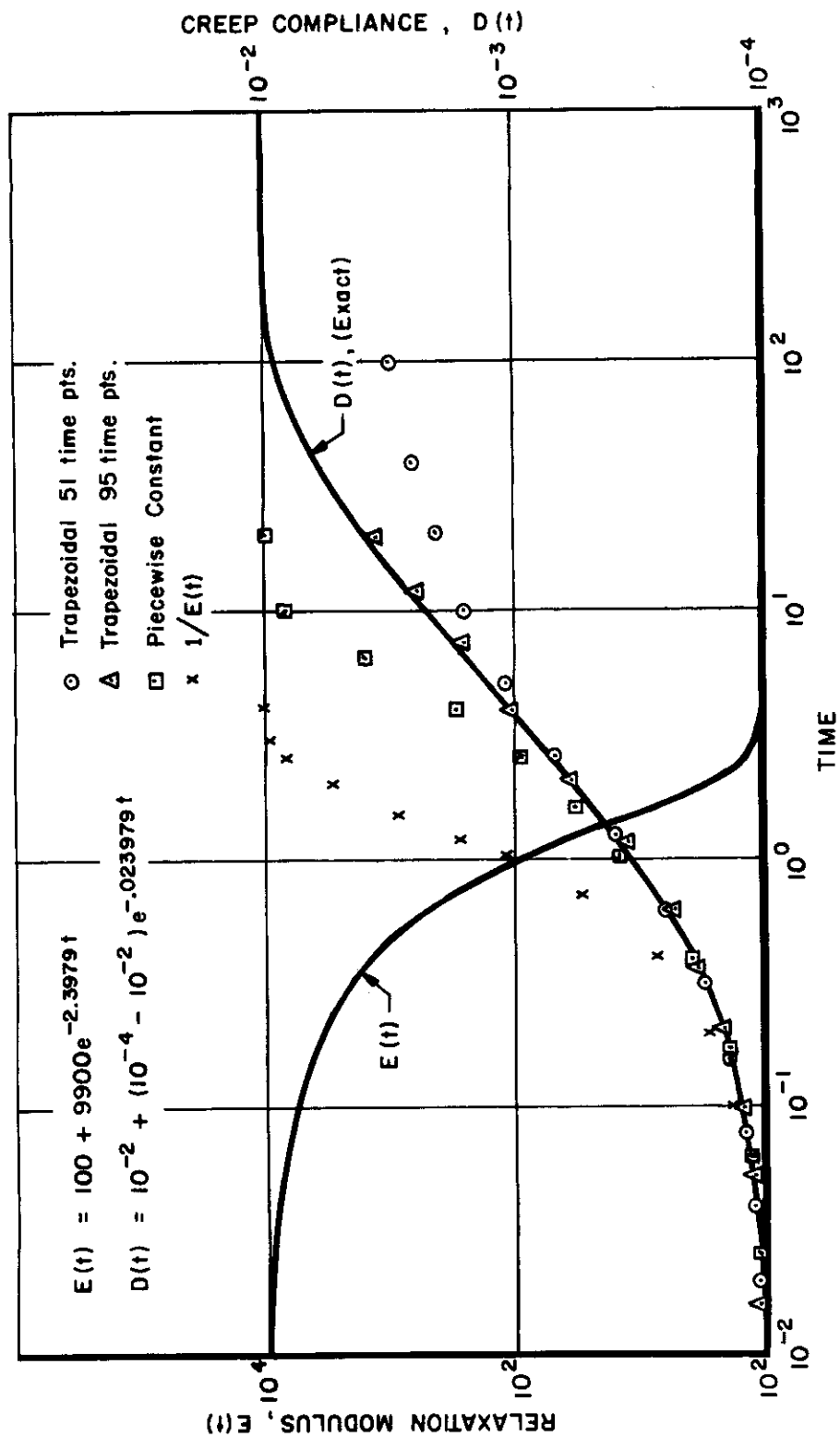


Figure 1. Numerical Inversion of the Relaxation Modulus

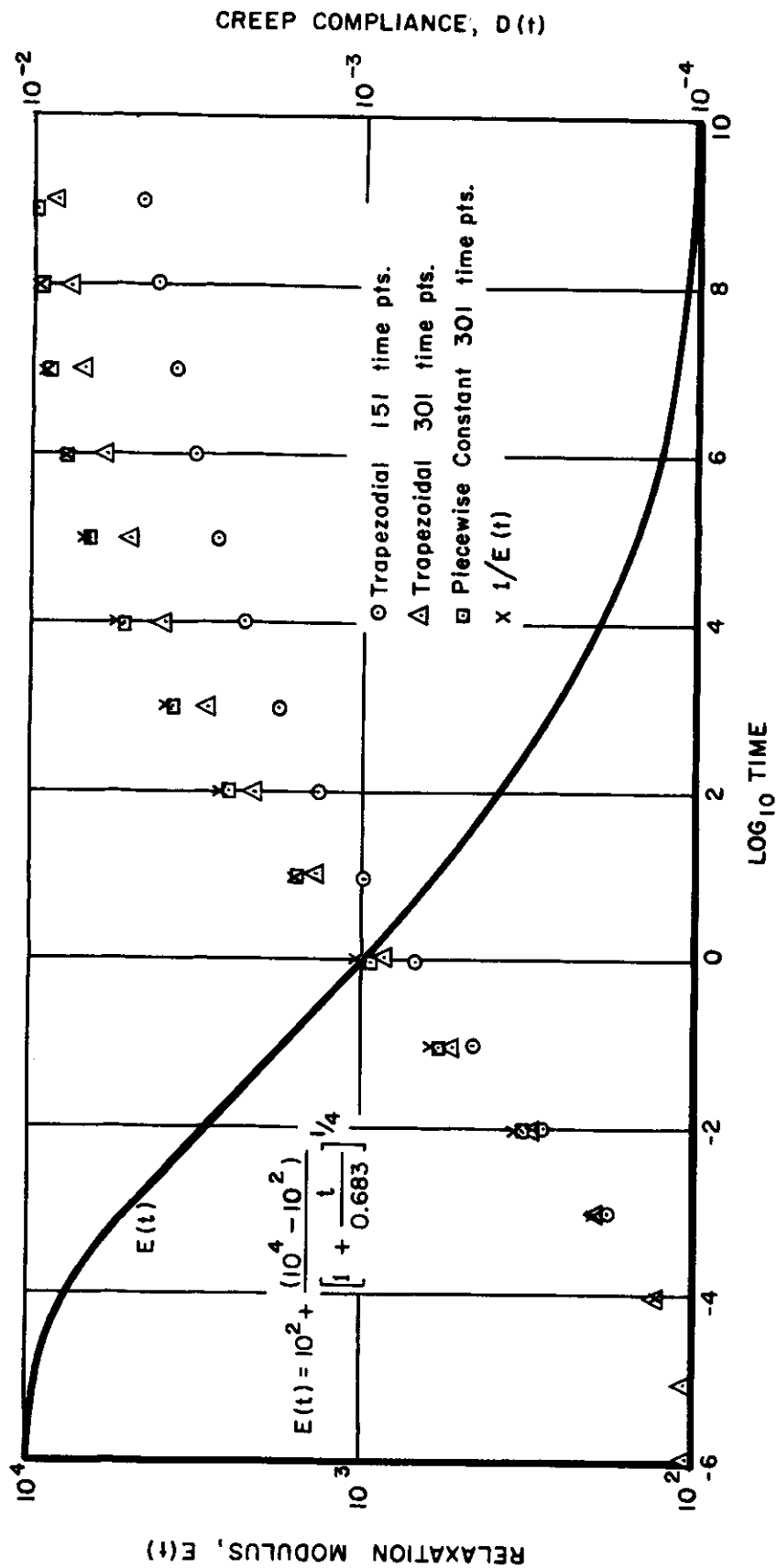


Figure 2. Numerical Inversion of the Relaxation Modulus



Interchanging the summation, the last term is written as

$$\sum_{j=1}^q A_j \sum_{i=1}^{k-2} e^{-\frac{t_k}{\tau_j}} \left( e^{\frac{t_{i+1}}{\tau_j}} - e^{\frac{t_i}{\tau_j}} \right) \epsilon^*(t_i) = \sum_{j=1}^q A_j \alpha_{j,k} \quad (6)$$

where

$$\alpha_{j,k} = e^{-\frac{t_k}{\tau_j}} \left( e^{\frac{t_{k-1}}{\tau_j}} - e^{\frac{t_{k-2}}{\tau_j}} \right) \epsilon^*(t_{k-2}) + e^{-\frac{(t_k - t_{k-1})}{\tau_j}} \sum_{i=1}^{k-3} e^{-\frac{t_{k-1}}{\tau_j}} \left( e^{\frac{t_{i+1}}{\tau_j}} - e^{\frac{t_i}{\tau_j}} \right) \epsilon^*(t_i) \quad (7)$$

or

$$\alpha_{j,k} = e^{-\frac{(t_k - t_{k-1})}{\tau_j}} \left[ \left( 1 - e^{-\frac{(t_{k-1} - t_{k-2})}{\tau_j}} \right) \epsilon^*(t_{k-2}) + \alpha_{j,k-1} \right] \quad (8)$$

The solution of Equation 1 may now be written

$$\epsilon(t_k) = \frac{1}{E(0) + E(t_k - t_{k-1})} \left[ \sigma(t_k) + \frac{E(0) - E(t_k - t_{k-1})}{2} \epsilon(t_{k-1}) + \sum_{j=1}^q A_j \alpha_{j,k} \right] \quad (9)$$

By using Zak's method we have exchanged the summation over the time realm for a summation over the Prony series terms plus a recurrence relation. Only the immediately past two solutions are now required to be saved to account for memory effects. The adequacy of Zak's method depends on how accurately the Prony series represents the relaxation modulus.

### SECTION III

#### FINITE ELEMENT DERIVATIONS

#### STRESS-STRAIN RELATIONS

The thermoviscoelasticity theory used in the finite element derivations has been presented elsewhere (Reference 10) and only the essentials are given here.

Assuming constant bulk modulus,  $K$ , and constant coefficient of thermal expansion,  $\alpha$ , the thermoviscoelastic constitutive relations may be written, using cartesian tensor notation, as

$$\begin{aligned} \sigma_{ij}(\mathbf{X}, t) = & 2G(o) \epsilon_{ij}(\mathbf{X}, t) - 2 \int_{o^+}^t \frac{\partial G(\xi - \xi')}{\partial t} \epsilon_{ij}(\mathbf{X}, t') dt' \\ & + \delta_{ij} \left[ K - \frac{2}{3} G(o) \right] \epsilon_{mm}(\mathbf{X}, t) + \delta_{ij} \frac{2}{3} \int_{o^+}^t \frac{\partial G(\xi - \xi')}{\partial t'} \epsilon_{mm}(\mathbf{X}, t') dt' \\ & - \delta_{ij} 3\alpha K [T(\mathbf{X}, t) - T_1(\mathbf{X}, o)] \end{aligned} \quad (10)$$

where allowance for an initial response has been made explicit,  $G(t)$  is the shear relaxation modulus given for an arbitrary "material reference temperature",  $T_o$ . The notation  $\mathbf{X}$  denotes spatial dependence. The shifted time  $\xi$  is related to real time  $t$  through the relation

$$\xi = \xi(\mathbf{X}, t) = \int_0^t \frac{dt'}{A_T[T(\mathbf{X}, t')]} \quad (11)$$

where  $T(\mathbf{X}, t)$  is temperature, while  $T_1(\mathbf{X}, o)$  = structural reference temperature, an initial steady-state temperature distribution with associated time constant stress and displacement states. The form to be used here for the shift function  $A_T(T)$  makes use of the so-called WLF equation (Reference 11)

$$\log_{10} A_T(T) = - \frac{C_1 (T - T_o)}{C_2 + (T - T_o)} = -h(T) \quad (12)$$

or

$$\frac{1}{A_T[T(\mathbf{X}, t)]} = 10^{h[T(\mathbf{X}, t)]} \quad (13)$$

where  $C_1$  and  $C_2$  are constants for a particular material. Upon the restriction to plane strain, Equation 10 becomes in matrix notation

$$\sigma(X, t) = L \epsilon(X, t) + C \epsilon(X, t) - 3\alpha I \Theta(X, t) \quad (14)$$

where

$$L \epsilon(X, t) = \frac{1}{3} \int_{0^+}^t \frac{\partial G}{\partial t'} (\xi - \xi') \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \epsilon_x(X, t') \\ \epsilon_y(X, t') \\ \gamma_y(X, t') \end{bmatrix} dt' \quad (15)$$

$$C = \begin{bmatrix} K + \frac{4}{3} G(0) & K - \frac{2}{3} G(0) & 0 \\ K - \frac{2}{3} G(0) & K + \frac{4}{3} G(0) & 0 \\ 0 & 0 & G(0) \end{bmatrix}$$

$$I = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Theta(X, t) = [T(X, t) - T_1(X, 0)]$$

One should be reminded that the law Equation 10 is nonlinear in temperature, thus thermal stress solutions cannot be superimposed. Since temperatures are prescribed this nonlinearity does not create any special problem.

#### ELEMENTAL EQUATIONS

Derivation of the elemental equations proceeds from the principle of virtual displacements in the form

$$\int_{t_1}^{t_2} \left( \int_{Vol} \delta \epsilon^T(X, t) \sigma(X, t) dV - \int_{Vol} \delta u^T(X, t) F_v(X, t) dV - \int_{S_T} \delta u^T(S, t) F_s(S, t) dS \right) dt = 0 \quad (16)$$

where

$\mathbf{F}_v(\mathbf{X}, t)$  = prescribed body force vector

$\mathbf{F}_s(\mathbf{S}, t)$  = prescribed surface tractions on portion of boundary,  $S_T$ .

$\delta \mathbf{u}(\mathbf{X}, t) = \begin{bmatrix} \delta u(x, y, t) \\ \delta v(x, y, t) \end{bmatrix}$  = x and y directed virtual displacements

$\mathbf{S}$  denotes the boundary position vector.

Linear displacements are assumed in each element. This leads to constant strains in each element giving the familiar expression

$$\begin{bmatrix} \epsilon_x(t) \\ \epsilon_y(t) \\ \gamma_{xy}(t) \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{bc} & 0 & y_{ca} & 0 & -y_{ba} & 0 \\ 0 & -x_{bc} & 0 & -x_{ca} & 0 & x_{ba} \\ -x_{bc} & y_{bc} & -x_{ca} & y_{ca} & x_{ba} & -y_{ba} \end{bmatrix} \begin{bmatrix} u_a(t) \\ v_a(t) \\ u_b(t) \\ v_b(t) \\ u_c(t) \\ v_c(t) \end{bmatrix}$$

or

$$\boldsymbol{\epsilon}(t) = \boldsymbol{\phi} \mathbf{q}(t) \quad (17)$$

where referring to Figure 3

$$y_{bc} = y_b - y_c, \quad x_{bc} = x_b - x_c, \text{ etc.}$$

A = element area

Linear distribution of temperature in each element is assumed, giving

$$\Theta(\mathbf{X}, t) = \begin{bmatrix} 1 & x & y \end{bmatrix} \mathbf{N} \begin{bmatrix} \Theta_a(t) \\ \Theta_b(t) \\ \Theta_c(t) \end{bmatrix} \quad (18)$$

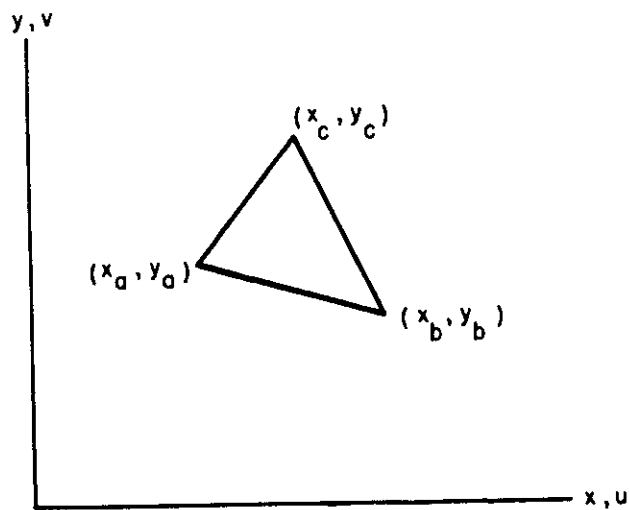


Figure 3. Typical Finite Element

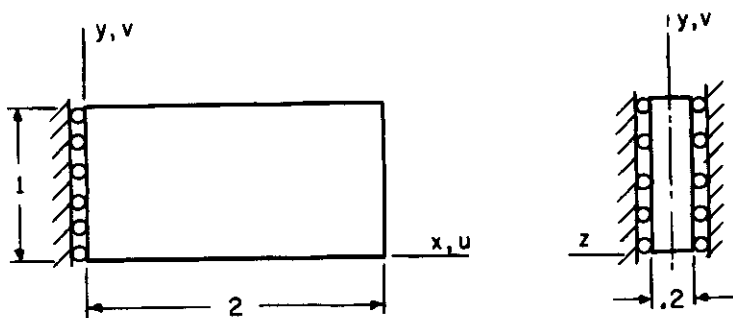


Figure 4. Geometry of Problem F-1, T-1

where again referring to Figure 3

$$\mathbf{N} = \frac{1}{2A} \begin{bmatrix} x_b y_c - x_c y_b & x_c y_a - x_a y_c & x_a y_b - x_b y_a \\ y_{bc} & y_{ca} & -y_{ba} \\ -x_{bc} & -x_{ca} & x_{ba} \end{bmatrix} \quad (19)$$

$\Theta_a(t)$ ,  $\Theta_b(t)$ ,  $\Theta_c(t)$  = nodal temperature changes.

The shifted time is calculated using an average nodal temperature in each element

$$\xi = \xi(t) = \int_0^t 10^{-h} [T_m(t')] dt' \quad (20)$$

where

$$T_m(t) = \left( \frac{T(x_a, t) + T(x_b, t) + T(x_c, t)}{3} \right) = \text{average nodal temperature for an element.}$$

Insertion of Equations 14, 17, and 18 into Equation 16 results in

$$\int_{t_1}^{t_2} \left( \delta \mathbf{q}^T(t) \boldsymbol{\phi}^T \left( \int_{Vol} \mathbf{L} \boldsymbol{\epsilon}(\mathbf{x}, t) + \mathbf{C} \boldsymbol{\phi} \mathbf{q}(t) - 3K\alpha \mathbf{I} \begin{bmatrix} 1 & x & y \end{bmatrix} \mathbf{N} \Theta(t) dV \right) \delta \mathbf{q}^T(t) \mathbf{F}(t) \right) dt = 0 \quad (21)$$

where the virtual work of body and external loads has been replaced by the virtual work of a statically equivalent set of nodal forces. Upon integration throughout the element volume, and assuming the  $\delta q_i(t)$  are independent,

$$\boldsymbol{\phi}^T \mathbf{L} \boldsymbol{\epsilon}(\mathbf{x}, t') + \boldsymbol{\phi}^T \mathbf{C} \boldsymbol{\phi} \mathbf{q}(t) - 3K\alpha \boldsymbol{\phi}^T \mathbf{I} \begin{bmatrix} 1 & x_m & y_m \end{bmatrix} \mathbf{N} \Theta(t) = \mathbf{F}(t) \quad (22)$$

where

$$x_m = \frac{x_a + x_b + x_c}{3} \quad y_m = \frac{y_a + y_b + y_c}{3}$$

The force vector,  $\mathbf{F}(t)$ , is to be interpreted as the forces per unit element thickness. The product  $3K \boldsymbol{\phi}^T \mathbf{I} \begin{bmatrix} 1 & x_m & y_m \end{bmatrix} \mathbf{N}$  is given the designation  $\mathbf{D}$ . Then

$$\mathbf{D} = \frac{3K\alpha}{2A} \begin{bmatrix} y_{bc} \\ -x_{bc} \\ y_{ca} \\ -x_{ca} \\ -y_{ba} \\ x_{ba} \end{bmatrix} [\mathbf{PQR}] \quad (23)$$

where

$$P = x_b y_c - x_c y_b + x_m y_{bc} - y_m x_{bc}$$

$$Q = x_c y_a - x_a y_c + x_m y_{ca} - y_m x_{ca}$$

$$R = x_a y_b - x_b y_a - x_m y_{ba} + y_m x_{ba}$$

Letting  $K_1 = \phi^T C \phi$ , the resulting expression

$$\phi^T L \epsilon(X, t) + K_1 q(t) - D \Theta(t) = F(t) \quad (24)$$

looks familiar except for the first term, which requires time integration from  $t' = 0$  to  $t' = t$ . This term (see the one dimensional example in the previous section) is approximated numerically as follows for the  $k$ th time point,

$$\begin{aligned} \phi^T L \epsilon(X, t) = & \phi^T \frac{1}{3} \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \left( \sum_{i=1}^{k-2} [G(\xi_k - \xi_{i+1}) - G(\xi_k - \xi_i)] \begin{bmatrix} \epsilon_x^*(t_i) \\ \epsilon_y^*(t_i) \\ \gamma_{xy}^*(t_i) \end{bmatrix} \right. \\ & + \frac{1}{2} [G(0) - G(\xi_k - \xi_{k-1})] \begin{bmatrix} \epsilon_x(t_{k-1}) \\ \epsilon_y(t_{k-1}) \\ \gamma_{xy}(t_{k-1}) \end{bmatrix} \\ & \left. + \frac{1}{2} [G(0) - G(\xi_k - \xi_{k-1})] \begin{bmatrix} \epsilon_x(t_k) \\ \epsilon_y(t_k) \\ \gamma_{xy}(t_k) \end{bmatrix} \right) \end{aligned} \quad (25)$$

where the average strain

$$\epsilon_x^*(t_i) = \frac{\epsilon_x(t_{i+1}) + \epsilon_x(t_i)}{2}, \text{ etc.}$$

All of the strains in Equation 25 are known except  $[\epsilon_x(t_k) \quad \epsilon_y(t_k) \quad \gamma_{xy}(t_k)]$ . Using Equation 17, we write

$$\phi^T L \epsilon(X, t') = -V(t_k) + [G(0) - G(\xi_k - \xi_{k-1})] K_2 q(t_k) \quad (26)$$

where

$$\begin{aligned} V(t_k) = & -M \left( \sum_{i=1}^{k-2} [G(\xi_k - \xi_{i+1}) - G(\xi_k - \xi_i)] \epsilon(t_i) \right. \\ & \left. + \frac{1}{2} [G(0) - G(\xi_k - \xi_{k-1})] \epsilon(t_{k-1}) \right) \end{aligned} \quad (27)$$

$$K_2(t_k) = \frac{1}{6} \phi^T \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \phi$$

and

$$M = \frac{1}{3} \phi^T \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

The governing incremental stiffness equations for an element are now

$$\underbrace{(K_1 + [G(o) - G(\xi_k - \xi_{k-1})])}_{\text{(Constant)}} \underbrace{K_2}_{\text{(Constant)}} q(t_k) = \underbrace{F(t_k)}_{\substack{\text{(Mechanical} \\ \text{Load)}}} + \underbrace{H(t_k)}_{\substack{\text{(Thermal} \\ \text{Load)}}} + \underbrace{V(t_k)}_{\substack{\text{(Memory} \\ \text{Load)}}} \quad (28)$$

The element stresses are written for the element centroid since temperature is permitted to vary linearly in each element.

$$\begin{bmatrix} \sigma_x(t_k) \\ \sigma_y(t_k) \\ \tau_{xy}(t_k) \end{bmatrix} = S_1 + S_2 \begin{bmatrix} \epsilon_x(t_k) \\ \epsilon_y(t_k) \\ \gamma_{xy}(t_k) \end{bmatrix} + S_3 \begin{bmatrix} \Theta_a(t_k) \\ \Theta_b(t_k) \\ \Theta_c(t_k) \end{bmatrix} \quad (29)$$

where

$$S_1 = \frac{1}{3} \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \left( \sum_{i=1}^{K-2} [G(\xi_k - \xi_{i+1}) - G(\xi_k - \xi_i)] \epsilon^*(t_i) \right. \\ \left. + \frac{1}{2} [G(o) - G(\xi_k - \xi_{k-1})] \epsilon(t_{k-1}) \right) = (\phi^T)^{-1} M V(t_k)$$

$$S_2 = \frac{1}{6} \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} [G(o) - G(\xi_k - \xi_{k-1})] + C$$

$$S_3 = -\frac{3K\alpha}{2A} \begin{bmatrix} P & Q & R \\ P & Q & R \\ 0 & 0 & 0 \end{bmatrix}$$



Using the development in the preceding section, the specialization of Equations 28 and 29 to Zak's method is accomplished by

$$\begin{aligned}
 \mathbf{V}(t_k) &= \mathbf{M} \left( \sum_{j=1}^q \mathbf{A}_j \mathbf{a}_{j,k} + \frac{1}{2} \left[ \mathbf{G}(0) - \mathbf{G}(\infty) - \sum_{j=1}^q \mathbf{A}_j e^{\frac{-(\xi_k - \xi_{k-1})}{\tau_j}} \right] \mathbf{e}(t_{k-1}) \right) \\
 \mathbf{a}_{j,k} &= e^{\frac{-(\xi_k - \xi_{k-1})}{\tau_j}} \left[ \left( 1 - e^{\frac{-(\xi_{k-1} - \xi_{k-2})}{\tau_j}} \right) \mathbf{e}^*(t_{k-2}) + \mathbf{a}_{j,k-1} \right] \\
 \mathbf{a}_{j,1} &= \mathbf{a}_{j,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
 \mathbf{a}_{j,3} &= e^{\frac{-\xi_3}{\tau_j}} \left( e^{\frac{\xi_2}{\tau_j}} - 1 \right) \mathbf{e}^*(0)
 \end{aligned} \tag{30}$$

One potential advantage of Equation 28 should be noted. Since  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are constant elemental matrices, for isothermal conditions and equal time increments, the merged stiffness matrix will remain constant from step to step. And for isothermal conditions and unequal time steps, a merged  $\mathbf{K}_1$  and merged  $\mathbf{K}_2$  can be stored for all time, thus eliminating regeneration and merging of these elemental matrices at each time point. Similar advantages may be possible for transient, homogeneous temperature distribution.

## SECTION IV

## THERMOVISCOELASTIC ANALYSIS PROGRAM (TAP) COMPUTER CODE

The TAP code was written as a pilot program to study the feasibility of our approach. Originally, out-of-core storage was provided for so that an unlimited number of solution time steps could be used. The need for frequent program changes coupled with excessive turn-around time made it advisable to write an in-core version which is presently the operable program. Coding is in FORTRAN IV for the UNIVAC 1108 computer. Zak's method is included as an option, although none of the results presented in the figures were obtained by this method.

As an indication of the program capacity and efficiency, the problem shown in Figure 10 for 91 time points took 9 minutes of computer time. Approximately 5 minutes of this time was spent in generating memory loads whereas using a 10-coefficient, Prony series with Zak's method, approximately one minute is necessary for generating memory loads. For this problem, approximately 100 time points could be normally used before exceeding core limitations whereas Zak's method, the number of time points is essentially unrestricted.

## SECTION V

### SOLUTION OF PROBLEMS

#### ELEMENTARY PROBLEMS

In order to evaluate the basic operation and accuracy of the TAP program, a problem with simple geometry was chosen for which exact analytical solutions could be obtained. The problem geometry is illustrated in Figure 4. The material properties chosen do not correspond to any particular material.

The first loading (F-1) considered was a unit step load of 10 on the right hand face. The exact solution is easily found to be

$$u(x, t) = \frac{5}{14} \cdot 10^{-3} \left[ 70.3494 H(t) - .04942 e^{-2.0588t} - 69.3000 e^{-.02398t} \right] x$$

The TAP results using four elements are shown in Figure 5 along with the corresponding finite element solutions by Schapery's direct method and the quasi-elastic method. The plotted results are a valid comparison between the three methods only to the degree that the material properties represent a realistic viscoelastic material. A transform parameter-time correspondence factor of  $p = \frac{1}{2t}$  was used in the Schapery method. In view of the form of the relaxation modulus curve in Figure 1 the quasi-elastic result was not unexpected.

Next considered was a temperature loading (T-1),

$$T(x, t) = H(t)(18 + 6x)(1 - .5 e^{-2t}),$$

with the structural and material reference temperatures taken as zero. Following essentially the procedure in Reference 10 the exact solution is:

$$\begin{aligned} \frac{u(x, t)}{\alpha} = & 1.2857 \int_0^x \left[ T(x', t) + .3391 e^{-(2.0588 \int_0^t 10^h [T(x', t')] dt')} \right. \\ & \left. \int_0^t e^{(2.0588 \int_0^{t'} 10^h [T(x', t'')] dt'')} \cdot T(x', t') \cdot 10^h [T(x', t')] dt' \right] dx \end{aligned} \quad (31)$$

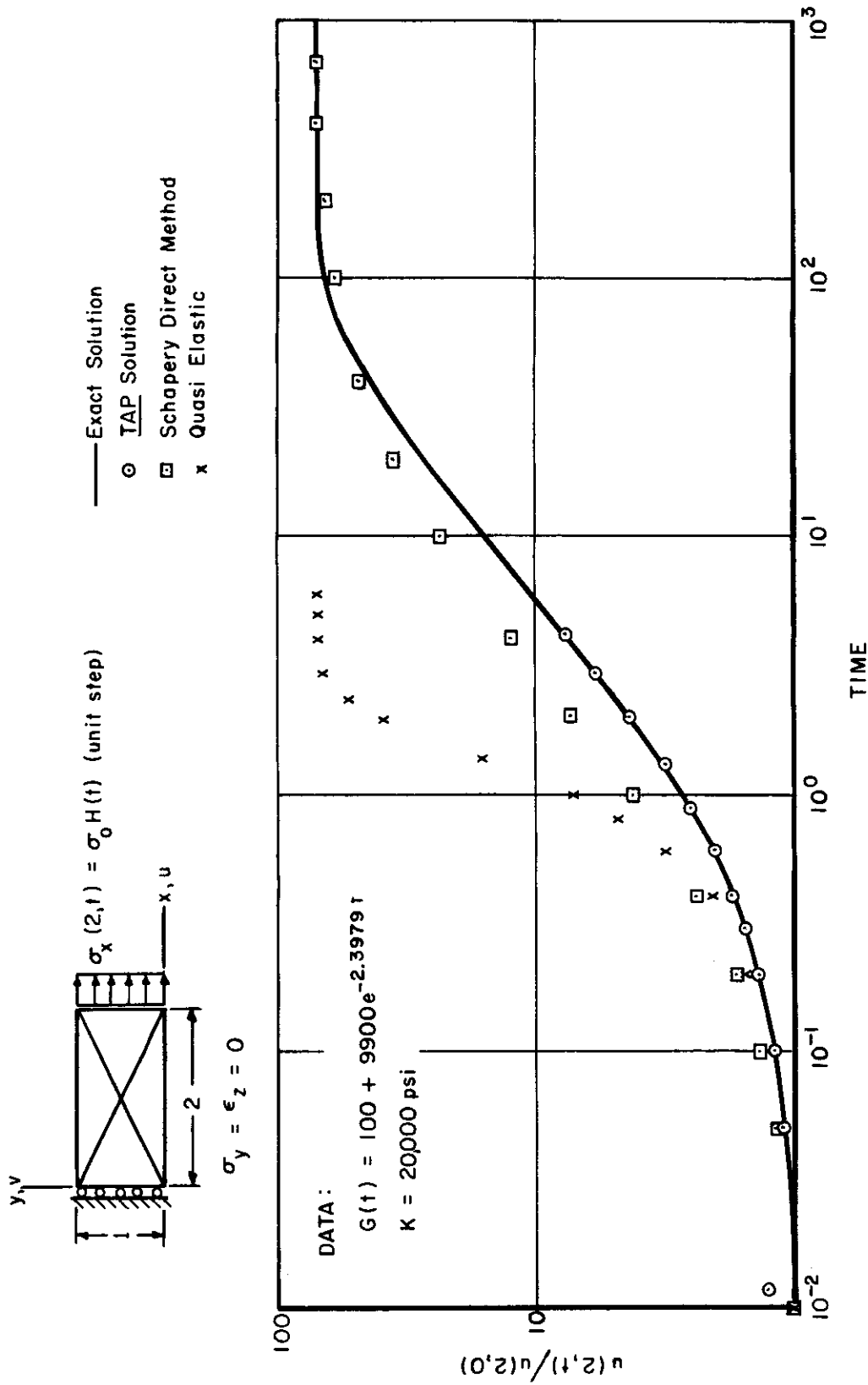


Figure 5. Elementary Problem, F-1

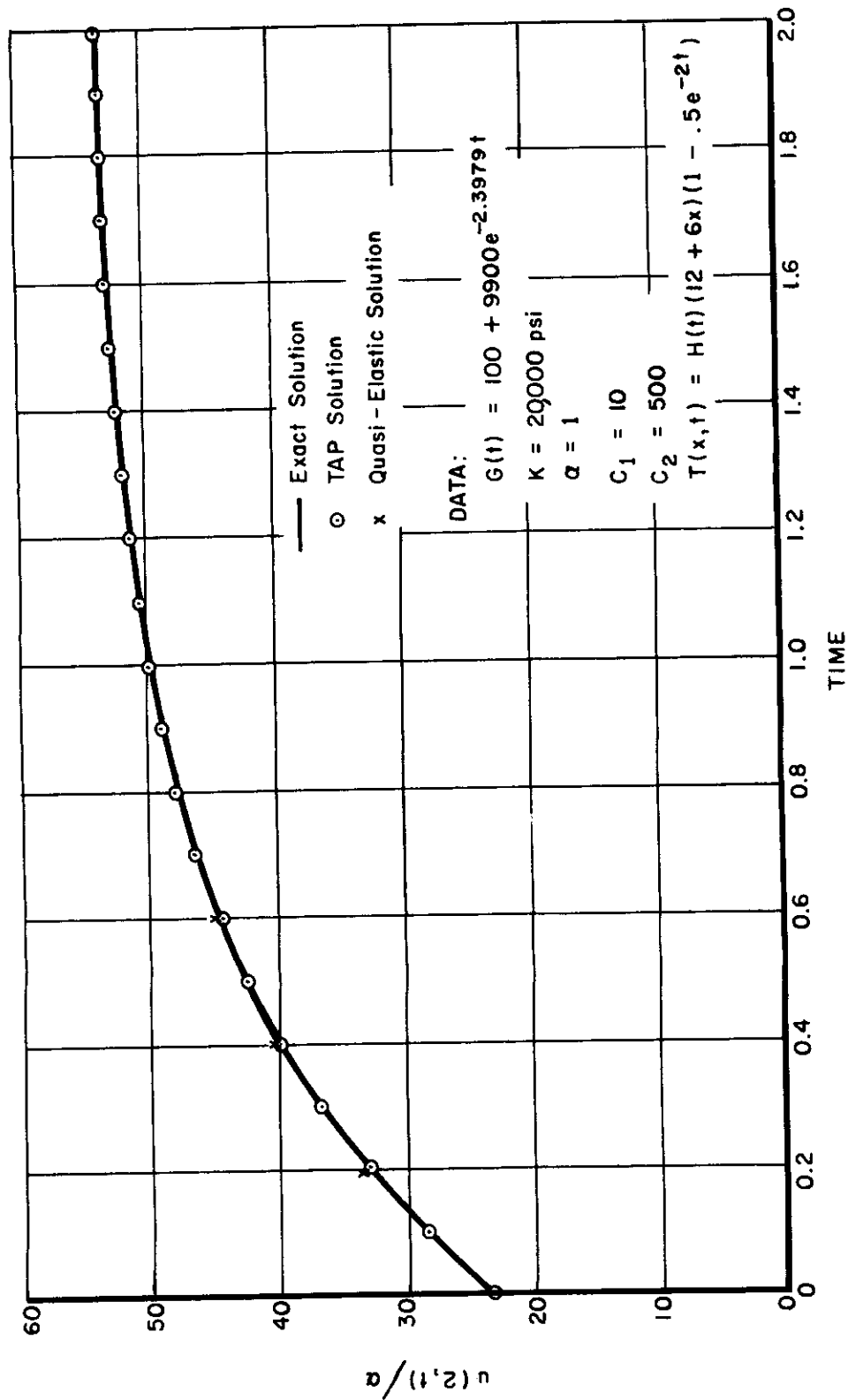


Figure 6. Elementary Problem, T-1

The four element TAP solution (Figure 6), using 0.1 time intervals, essentially coincided with the exact solution. The quasi-elastic, finite element solution was nearly exact also which is not surprising in view of the different type of loading and loading rate in problem T-2.

## COMPLEX PROBLEMS

Two problems were chosen representing practical, solid rocket grain configurations, Figures 7 and 8. The elemental breakdowns are obviously not fine enough for a realistic stress analysis, but are sufficient for evaluating our methods.

The first problem (F-2) is an end burning grain, partially bonded circumferentially. Figure 7 shows a one-sixth segment of the cross section, perpendicular to the motor axis. The loading represents a 70°F, firing condition for which a step pressure and step radial displacement is applied over the unbonded and bonded portions respectively. Using experimental data in Figure 8, a shear modulus,  $G(t) = 1/3 E(t)$ , was assumed. Equal increments of log time,  $\Delta \log_{10} t = 1$ , were used. Computer running time was approximately 2.5 minutes. Each of the quasi-elastic solutions required 20 seconds. The TAP results, compared with quasi-elastic solutions in figure 7 demonstrate the inadequacy of the quasi-elastic method for this particular problem.

The second problem was a slotted grain configuration. Figure 10 shows a one-quarter segment of the cross section perpendicular to the motor axis. Assumptions include a rigid outer boundary, a uniform structural reference temperature  $T_1 = 70^\circ\text{F}$ , and a shear relaxation modulus shown in Figure 2. Time-temperature shift data from Figure 9 determined  $C_1$  and  $C_2$ . A temperature (T-2), representing cooling of the outer surface from 70°F to -70°F was first input as follows

$$T(r,t) = 70 - 70 (1 - \cos \pi / 5t)(r/8)^3 \quad r = \text{radial location}$$

the results for which are shown in Figure 10. The quasi-elastic solution, a commonly used method, agrees reasonably well for displacements but deviates greatly from the TAP solution for stress. The TAP results appear to converge with progressively finer time increments as indicated by the tabulated solutions in Figure 10.

A slower rate of cooling was considered finally as follows:

$$T(r,t) = 70 - 70 (1 - \cos \pi / 60t)(r/8)^3$$

It was anticipated that the quasi-elastic stress solution would be closer to the TAP results. Such was not the case as shown in Figure 11.

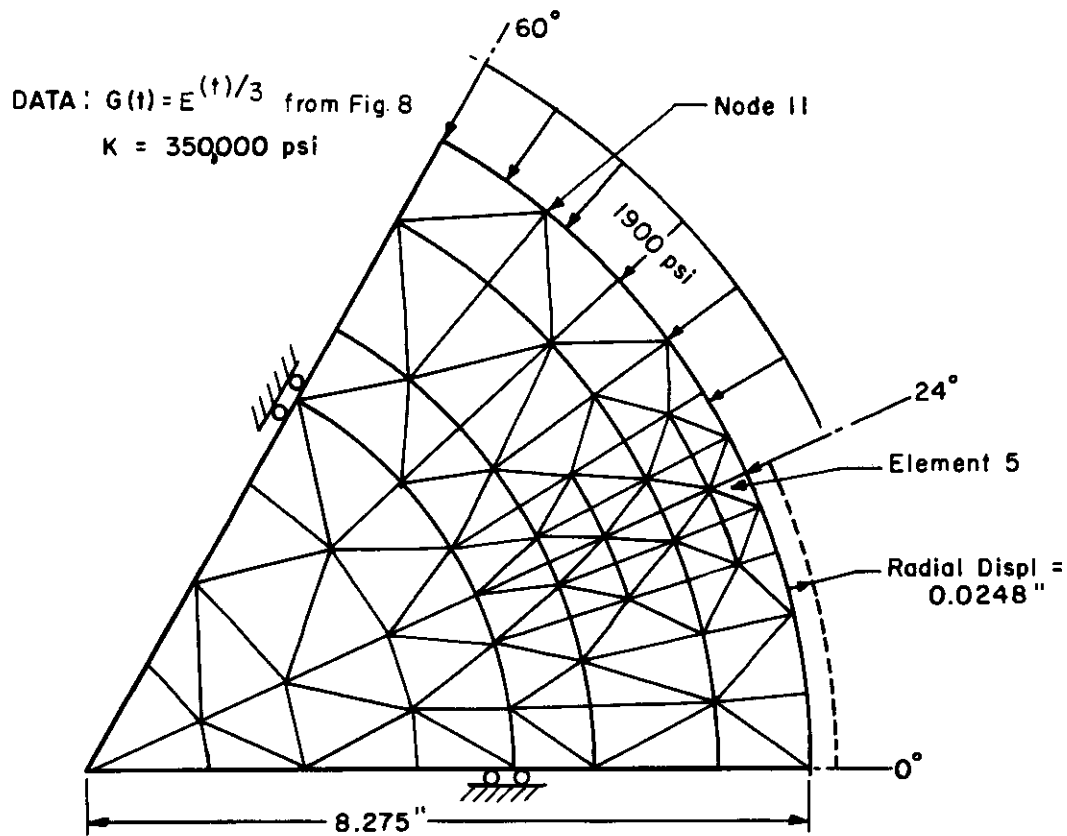
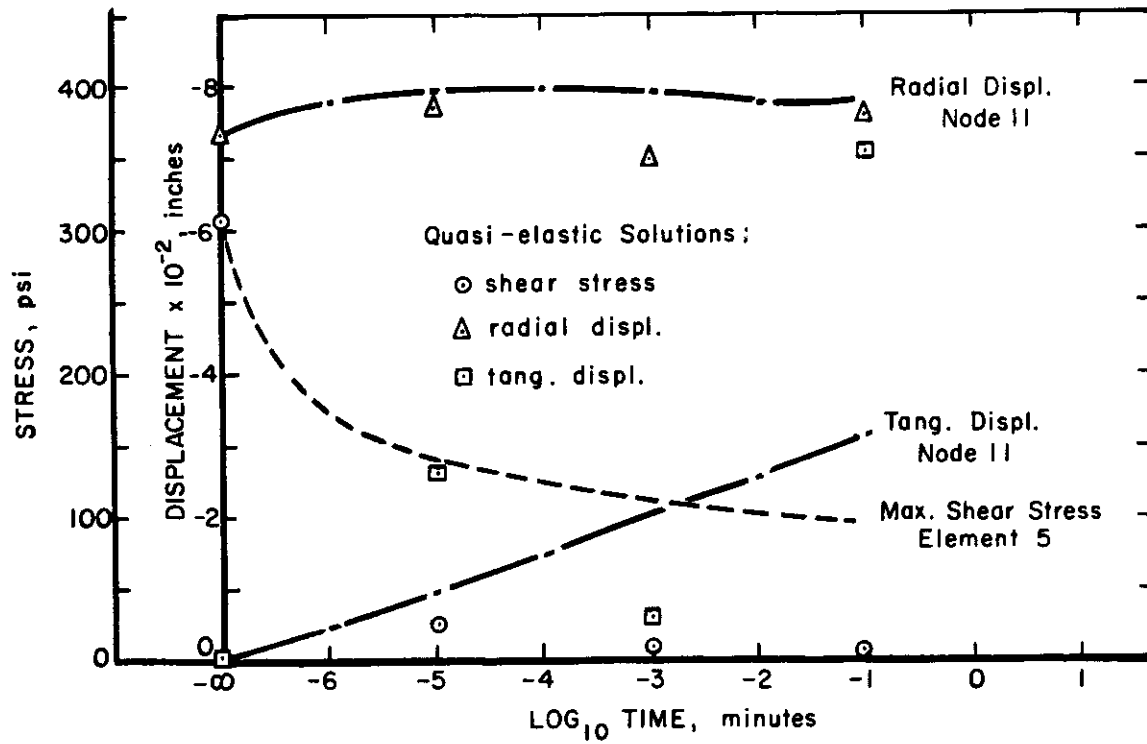


Figure 7. Problem F-2

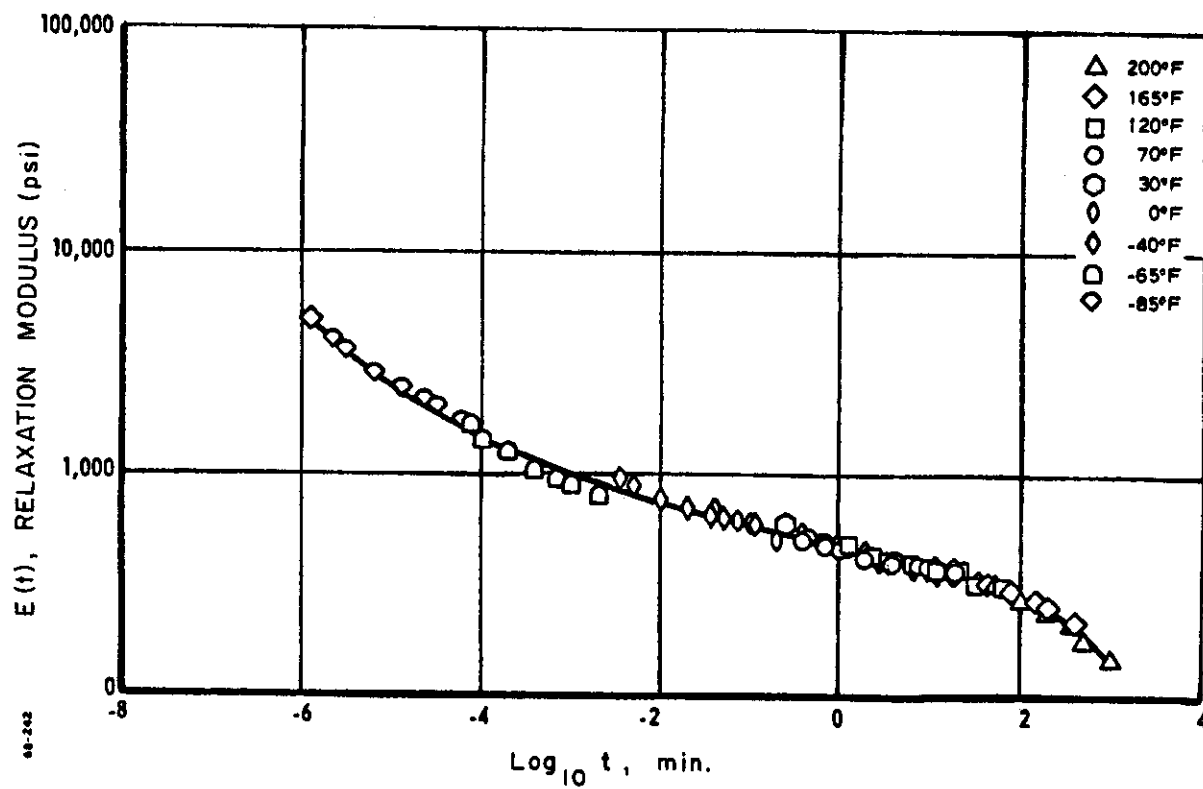


Figure 8. Relaxation Data for Problem F-2



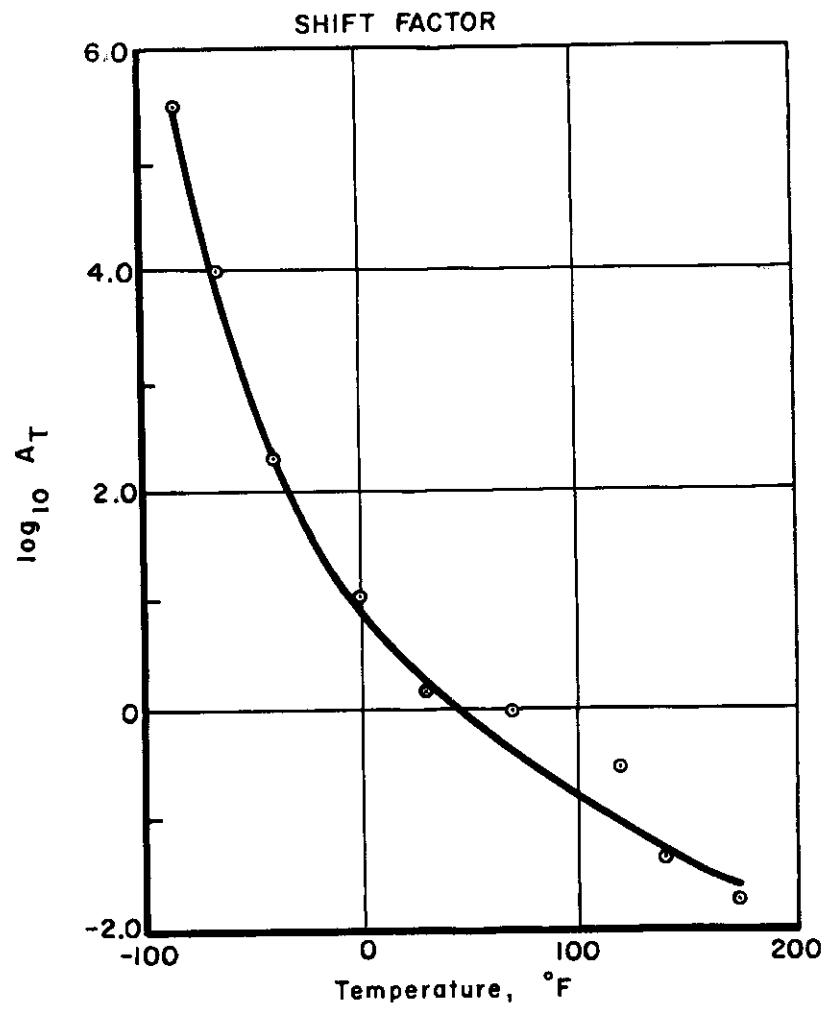


Figure 9. Time-Temperature Shift Data for Problem T-2.

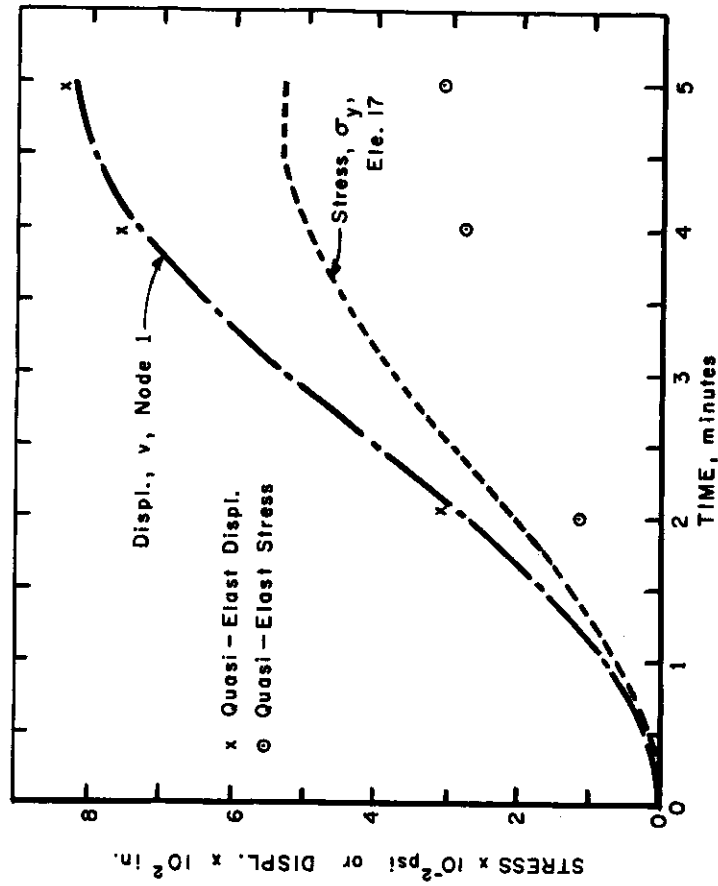
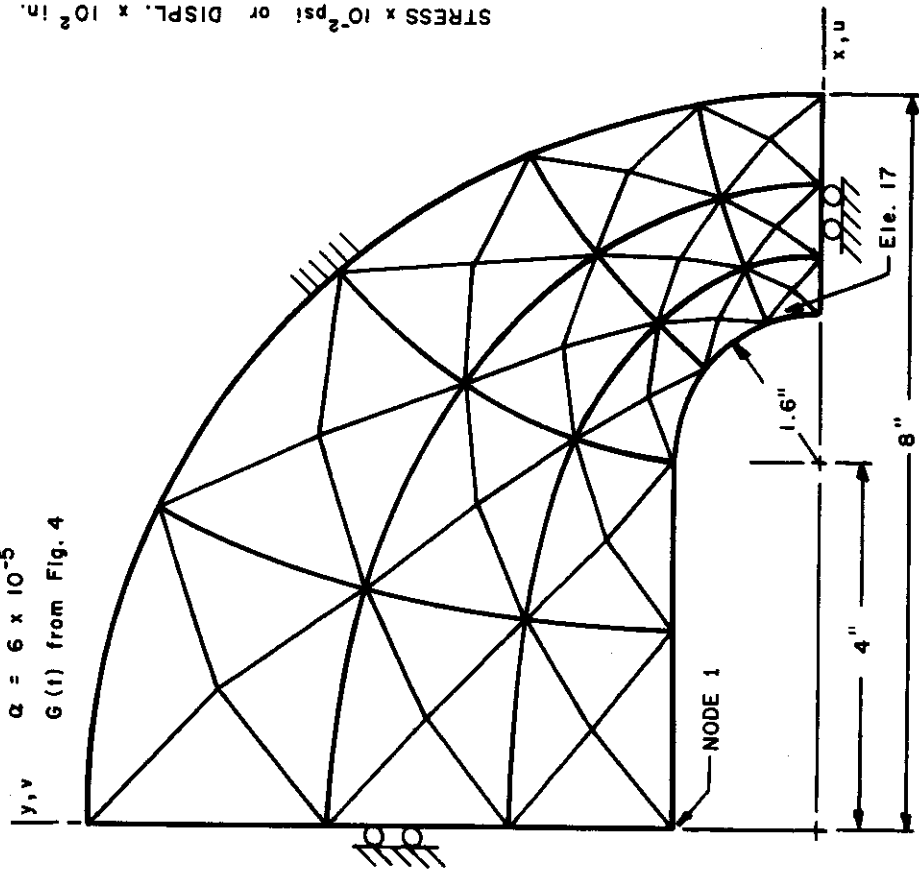
Temperature Input :  $T(r,t) = 70 - 70(1 - \cos \pi / 5^t)(r/8)^3$   
Struct. Ref. Temp :  $T_s = 70^\circ \text{F}$

$K = 100,000 \text{ psi}$

$C_1 = 3.05 \quad C_2 = 255.7$

$\alpha = 6 \times 10^{-5}$

$G(t)$  from Fig. 4



54 time pt. $\Delta t = 0.1$			91 time pt. $\Delta t = 0.05$		
TIME	$\bar{v}$	$\bar{\sigma}_y$	$\bar{v}$	$\bar{\sigma}_y$	
1	.0073	60.3	.0074	58.9	
2	.0272	206.6	.0274	202.3	
3	.0522	378.6	.0525	372.4	
4	.0730	500.3	.0734	494.0	
4.5	.0791	528.2	.0796	522.3	$t_{\max}$
5	.0817	527.2			$t_{\max}$

Figure 10. Slotted Grain Thermal Problem T-2

Temperature input:  $T(r,t) = 70 - 70(1 - \cos \frac{\pi}{60} t)(\frac{r}{8})^3$

Struct. Ref. Temp.:  $T_1 = 70^\circ\text{F}$

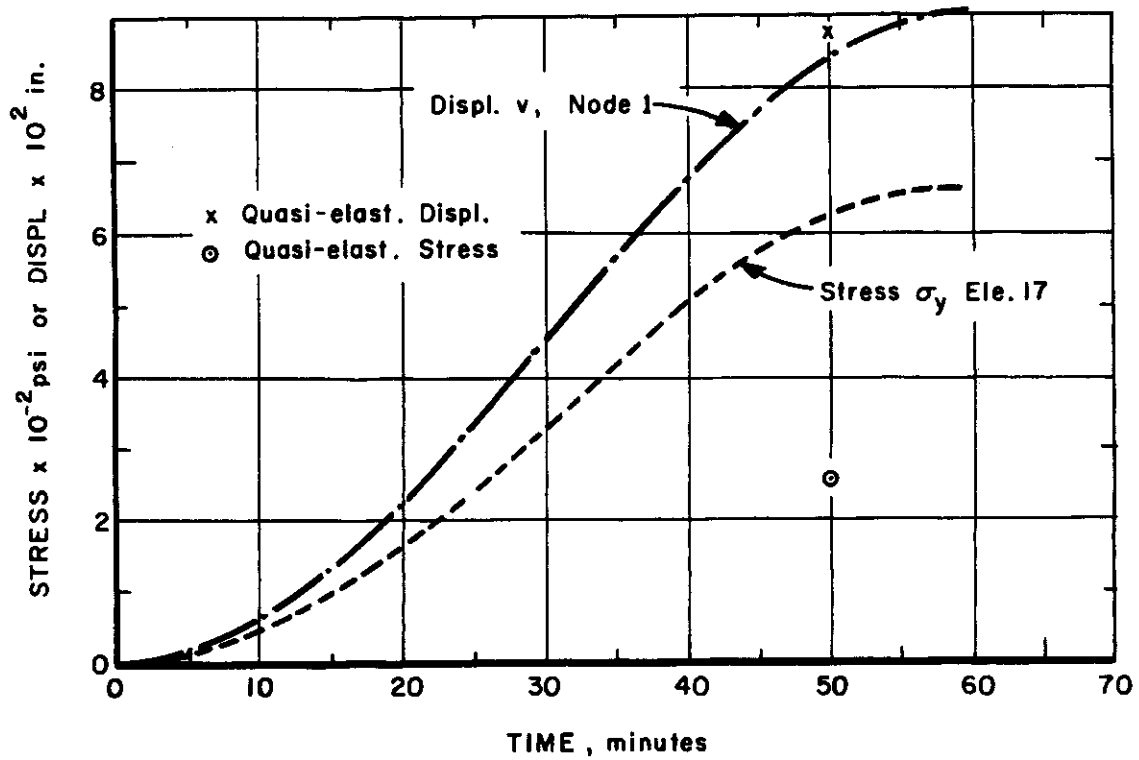


Figure 11. Slotted Grain Thermal Problem T-2A

## SECTION VI

### CONCLUSIONS

The two primary questions under study are: Within the assumptions of linear, quasi-static viscoelasticity, what is the adequacy of the presently used, approximate viscoelastic analysis methods? Is the direct formulation here using a finite element procedure a practical approach to viscoelastic stress analysis?

To definitively answer the first question, considerably more study is required. In particular, thermal cycling problems and mechanical loading under thermal environment should be studied for carefully chosen geometric configurations. Obviously, the question must be rephrased if real materials exhibit significant non-linearities, as appears to be the case for composite solid propellants for example.

The answer to the second question appears to be in the affirmative. In regard to computer time requirements, problem T-2 for 54 time points was extrapolated to the case of 500 unconstrained freedoms (versus 60 in T-2). Based on the TAP code, a rough estimate of 2 hours computer time would be required--using an out-of-core program of course. Using Zak's method in an in-core program, with 10 Prony coefficients, approximately 45 minutes would be required. For 91 time points the corresponding figures are 3 1/2 hours and with Zak's method, 80 minutes. Significant improvement could be expected with a more efficient code.

For short solution time realms, as in problem F-2, computer time would not generally be prohibitive for realistic, elemental breakdowns. Of significance in this case is the practicality of using directly, measured relaxation data rather than resorting to additional approximations.

## SECTION VII

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